Math 260: Linear Algebra
Proofs: Proving "For All" Statements

Prove or disprove:

1) $\forall\ x \in [3,5]$, $4x - 1 \in [11,19]$.

Proof: Let $x \in [3,5]$. Then $3 \leq x \leq 5$.

Multiply through by 4 $\Rightarrow$ $12 \leq 4x \leq 20$

Subtract 1 from all parts $\Rightarrow$ $11 \leq 4x-1 \leq 19$

So $4x-1 \in [11,19]$ ✓

Prove or disprove:

2) $\forall\ x \in \mathbb{R}$, if $\cos x = 0$, then $\sin 2x = 0$.

Proof: Let $x \in \mathbb{R}$ such that $\cos x = 0$.

Then $\sin 2x = 2 \sin x \cos x$

$= 2(\sin x)(0)$

$= 0$

So $\sin 2x = 0$ ✓

Prove or disprove:

3) For all integers n, $n^3 + 3n^2 + 2n + 5$ is odd.

proof: Let n be an integer.

Case 1: n is odd.

Then $n = 2k+1$ for some integer k.

Then $n^3+3n^2+2n+5 = (2k+1)^3 + 3(2k+1)^2 + 2(2k+1) + 5$

$= 8k^3 + 12k^2 + 6k + 1 + 12k^2 + 12k + 3 + 4k + 2 + 5$

$= 8k^3 + 24k^2 + 22k + 10 + 1$

$= 2[4k^3 + 12k^2 + 11k + 5] + 1$

$= 2m + 1$ which is odd

($m = 4k^3 + 12k^2 + 11k + 5$ is an integer because the integers are closed under addition and multiplication)

Case 2: n is even.

Then $n = 2k$ for some integer k.

Then $n^3+3n^2+2n+5 = (2k)^3 + 3(2k)^2 + 2(2k) + 5$

$= 8k^3 + 12k^2 + 4k + 4 + 1$

$= 2[4k^3 + 6k^2 + 2k + 2] + 1$

$= 2m + 1$ which is odd

($m = 4k^3 + 6k^2 + 2k + 2$ is an integer because integers are closed under addition and multiplication)

In any case, $n^3 + 3n^2 + 2n + 5$ is odd ✓

Prove or disprove:

4) $\forall\ x \in \mathbb{R}$, if $x \notin [-1,1]$, then $x(1-x) < 0$.

proof: Let $x \in \mathbb{R}$ such that $x \notin [-1,1]$.

Case 1: $x < -1$.

Then x is negative.

Also, multiply both sides of $x < -1$ by $-1 \Rightarrow -x > 1$

Now add 1 to both sides $\Rightarrow 1-x > 2$

So $1-x$ is positive.

Then $x(1-x) =$ (negative)(positive) $=$ negative

So $x(1-x) < 0$.

Case 2: $x > 1$.

Then x is positive.

Multiply both sides of $x > 1$ by $-1 \Rightarrow -x < -1$

Now add 1 to both sides $\Rightarrow 1-x < 0$

So $1-x$ is negative.

Then $x(1-x) =$ (positive)(negative) $=$ negative

So $x(1-x) < 0$.

In any case, $x(1-x) < 0$ ✓

Prove or disprove:

5) $\forall\ x \in \mathbb{R}\,,\ 2x^2 - 16x + 31 > 0.$

Counterexample $x = 4$

$x = 4 \in \mathbb{R}$, but $2x^2 - 16x + 31 = 2(4)^2 - 16(4) + 31$

$= -1 \not> 0.$

Prove or disprove:

6) For all $n \in \mathbb{N}$, $\frac{4n^2+4n}{2^n}$ is a natural number.

Counterexample $n=5$.

$n=5 \in \mathbb{N}$, but $\dfrac{4n^2+4n}{2^n} = \dfrac{4(5)^2+4(5)}{2^5}$

$= \dfrac{120}{32}$

$= \dfrac{15}{4}$ which is not a natural number.